Gauss's law

(B.Sc. (Physics) Part-II, Paper-IV, Group-A)

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(Dated: July 18, 2020)

"But still try, for who knows what is possible?"

— Michael Faraday (1791-1867)

Coulomb's law for a point charge q is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} q \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{1}$$

Note that we can define $|\mathbf{x}-\mathbf{x}'| = r$ for simplicity of notation. If we put the charge at the origin, we obtain

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \ . \tag{2}$$

In the above equation we dot both sides with the unit normal $\hat{\mathbf{n}}$ and obtain

$$\begin{aligned} \mathbf{E}\cdot\hat{\mathbf{n}} &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{n}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta \end{aligned} \tag{3}$$

Now perform the surface integral over the closed surface

$$\oint \mathbf{E}\cdot\hat{\mathbf{n}}\, da = \oint \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta\, da \tag{4}$$

Now using the relation $\cos\theta\, da = r^2\, d\Omega$ (using solid angle definition), we write the above

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equation as

$$\begin{aligned}\oint \mathbf{E}\cdot\hat{\mathbf{n}}\,da &= \oint \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}r^2\,d\Omega \\ &= \oint \frac{1}{4\pi\epsilon_0}q\,d\Omega \\ &= \frac{1}{4\pi\epsilon_0}q\oint\,d\Omega \\ &= \frac{1}{4\pi\epsilon_0}q4\pi \\ &= \frac{q}{\epsilon_0} \qquad (5)\end{aligned}$$

or

$$\boxed{\oint \mathbf{E}\cdot\,d\mathbf{a} = \frac{q}{\epsilon_0}}\,. \qquad (6)$$

This is the Gauss's law for a point charge placed at the origin. If we consider many charges scattered about instead of a single charge, we can write using the superposition principle or

$$\mathbf{E} = \sum_{i=1}^{n}\mathbf{E}_i. \qquad (7)$$

Now Eq. (6) for many charges $\{q_i\}$ for $i = 1, ..., n$, enclosed within the surface can be written as

$$\oint \mathbf{E}\cdot\,d\mathbf{a} = \sum_{i=1}^{n}\oint \mathbf{E}_i\cdot\,d\mathbf{a} = \sum_{i=1}^{n}\frac{q_i}{\epsilon_0} \qquad (8)$$

or

$$\boxed{\oint \mathbf{E}\cdot\,d\mathbf{a} = \frac{Q_{inc}}{\epsilon_0}}\,, \qquad (9)$$

where Q_{inc} denotes the total charge enclosed within the surface. For continuous charge distribution, $\mathbf{E}$ will be given by

$$\begin{aligned}\oint \mathbf{E}\cdot\,d\mathbf{a} &= \frac{1}{\epsilon_0}\int dq \\ &= \frac{1}{\epsilon_0}\int \frac{dq}{d\mathbf{x}}d\mathbf{x} \qquad (d\mathbf{x}\equiv dx\,dy\,dz)\end{aligned}$$

or

$$\oint \mathbf{E}\cdot\,d\mathbf{a} = \frac{1}{\epsilon_0}\int \rho(\mathbf{x})\,d\mathbf{x} \qquad (10)$$

Next, using divergence theorem [∗]

$$\oint_s \mathbf{V} \cdot \mathbf{da} = \int_v \nabla \cdot \mathbf{V} \, d\tau \ , \qquad\qquad (d\tau \equiv dx\, dy\, dz) \tag{11}$$

we can write Eq. (10) as

$$\int \nabla \cdot \mathbf{E} \, d\mathbf{x} = \frac{1}{\epsilon_0} \int \rho(\mathbf{x}) \, d\mathbf{x}$$

or

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{x})}{\epsilon_0}} \ . \tag{12}$$

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[1] D.J. Griffiths, *Introduction to Electrodynamics*, New Jersey: Prentice Hall, (1999) .

[2] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, (1962).

[∗] For details of divergence theorem, see B.Sc. Part-III Mathematical Physics notes uploaded on the website